Dynamic Survival Bias in Optimal Stopping Problems

Wanyi Chen *

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Abstract

This paper studies the optimal inference from observing an ongoing experiment. An experimenter is fully informed and sequentially chooses whether to continue with the costly trials that yield random payoffs. My twist is that outside observers see only the recent trial results, and not the earlier prehistory. I contrast the optimal sophisticated posterior based on a full Bayesian inference that accounts for the prehistory and the naive posterior based solely on the observed history. The resulting dynamic bias grows with longer prehistory if we see enough early successes. Observing more failures may increase the sophisticated posterior if they come early. Seeing previous successes (failures) always increases (lowers) the sophisticated posterior, but seeing future failure may increase the sophisticated posterior.

Keywords: Optimal stopping; Bayesian learning; Survival bias

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1 Introduction

In many economic scenarios, an outside observer learns about an underlying random variable from an insider's experiment. For instance, venture capitalists assess a firm through its market performance and financial status. Education institutions evaluate students' abilities through their exam scores. The signals observed by the observer also affect the insider's self-selection. A firm may exit the market if it learns that its capacity is low (Miklós-Thal et al. (2018)). Students may drop out if they expect poor performance in the future (Arcidiacono et al. (2016)).

I model an insider (he)'s self-selection as an optimal stopping problem. He undertakes a sequence of trials revealing a random payoff of interest and decides whether to stop experimenting after each trial. He continues only when the full history of signal realizations is sufficiently encouraging. This subsumes two typical classes of optimal stopping problems: (1) The one-armed bandit problem (Gittins (1979)), where the insider pays a flow cost to conduct the experiment and gathers the realization of each trial as rewards; (2) the optimal experimentation problem (as formulated in Moscarini and Smith (2001) as a special case for the Wald (1947)'s sequential decision problem), where the reward is realized only in the final period.

The focus of this paper is instead on the observer (we)'s belief about the underlying random variable. If we observe the full history of the signal realizations of the insider, it is a simple Bayesian updating exercise. Often, however, we see only the more recent signal realizations (the public history), but not the historical ones (the prehistory). I take as a benchmark a *naive posterior* that arises from Bayesian updating based solely on the public history. This naive measure fail to extrapolate the information content of the insider's action and results in a selection bias. The insider's stopping problem creates a more likely pattern of early good signal and later poor signals in the full history. Hence, our observation being the later truncation in the full history is a non-representative sample.

To correct this bias, I characterize a sophisticated posterior that considers all possible prehistories consistent with the public history. The public history critically shapes the inference of the prehistory, making the bias dynamic. This is reflected in the path-dependency of the sophisticated posterior, where earlier bad signals drive up the sophisticated posterior. Intuitively, early bad signals are associated to better unobserved prehistories, as it is less likely for the insider to continue experimenting after a streak of bad signals. The ordering of signals could outweigh their relative numbers.

The timing of the observation is also critical. When the public history contains enough early good signals, the bias accumulates as the "survival time" grows. While this conclusion fails when the public history contains many early failures, in which case we are more sure that the prehistory is good so that the insider keeps going if the prehistory is shorter. The uncertainty could compensate for the positive effect that the prehistory survived more stopping decisions. Moreover, the bias evolves during the observation. Regard the observation as a running process and consider the marginal effect of seeing one more signal realization. While observing more following good signals always increases the sophisticated posterior, the opposite is not always true. Note that we see not only additional signal realizations, but also the fact that the insider continues for one more period. This effect can dominate that of more bad signals. On the other hand, uncovering successes (resp. failures) in the prehistory always raises (resp. lowers) the sophisticated posterior. In addition, revealing a signal in either the recent or old prehistory has the same effect on the sophisticated posterior.

While the naive posterior suffers many flaws, its applications prevail and result in concrete consequences. For instance, the admission decisions in many universities[1] rely only on a one-time national exam. This policy mimics the naive posterior that ignores the prehistory of students' previous performance, which is critical in their optimal stopping problems. Students from low income families are more prone to drop out if their historical performance was unsatisfying. Ignoring the statistical bias creates systematic inequality. Unintentional IPO underpricing[2] may also reflect the bias that the underwriter does not fully consider the firm's previous experimentation.

For illustrative purposes, I proceed in the simplest possible setting of binary outcome gambles. The binary assumption allows us to specify the pattern of a sequence of signals, which plays a critical role in shaping our sophisticated posterior. But the intuition of all the results applies to general signal structures.

This model can be viewed as the learning of a single agent with limited memory. It is supported in many psychology literatures that recent memories are more frequently sampled when people make judgments (Tversky and Kahneman (1973)). The naive posterior represents the extreme case of accounting only for the recent memory. This insider's problem also extends to a more general sequential decisions, where the hidden signals must satisfy a censoring rule through out its path. For example, we evaluate a relatively new mutual fund by observing its yields. The fund manager must track the performance of the portfolio before launching the product. This prehistory is hidden from the investors who do not know the exact asset allocation. Or consider consumers assessing the effectiveness of a device, which must be tested before put into the market. This setting also accommodates the case where the insider and observer learn from different but correlated signals. For instance, drug firms conduct private experiments before applying for FDA trials. But the physician labeling[3] is based only on the FDA data, which could lead to conservative drug adoption.

[1] Typically, universities in China, Korea, and Turkey, see https://en.wikipedia.org/wiki/National_College_Entrance_Examination

[2] See Ercan et al. (2011) for a survey on IPO underpricing.

[3] https://www.accessdata.fda.gov/scripts/sda/sdNavigation.cfm?sd=labelingdatabase

Related Literature Selection bias is well documented in econometrics since Heckman (1979). There is also a large literature that studies biased beliefs in observational learning (as formulated in Banerjee (1992) and Bikhchandani et al. (1992)). Benjamin (2019) presents a comprehensive survey on this topic. Among the recent papers, Jehiel (2018) show that considering only undertaken projects leads to over-optimistic assessment. Esponda and Pouzo (2017) study voters' biased belief from missing information of unelected parties. Esponda (2008) introduces an adverse selection model with naive players who ignore the selection problem induced by their actions. The biases studied in these papers are typical survival biases where only signals from surviving processes are sampled. Moving beyond this setting, I focus on the signals from a single process with a dynamic selection by the insider's optimal stopping problem.

In literature on auto correlation, Miller and Sanjurjo (2018) provides a statistical explanation of the hot hand fallacy. He (2018) considers the mislearning from the gambler's fallacy with endogenously censored data. Papers on Berkson's bias (Berkson (1946)) considers the auto correlation from selection problems. My paper differs from existing literature of this branch in considering the serial correlation caused by the experimenter's sequential decisions.

This paper also relates to learning with bounded memory in statistics. Literatures following Cover and Hellman (1970) develop the optimal sequential hypothesis testing algorithm in various environments. Leighton and Rivest (1986) finds the optimal memory state to estimate the probability of a Bernoulli process. Wilson (2014) studies the confirmation bias due to bounded memory. These papers, however, do not explicitly analyze the posterior and the relevant comparative statistics due to technical difficulties. The memory state in my model is fixed, making the posterior tractable.

Some of my results are reflected in the literature of strategic experimentation. Bolton and Harris (1999) first introduce the multi-agent bandit problem where experimenters also observe signals from others' experiments. Dong (2018) extends this model by considering asymmetric prior. These papers focus on the behavioral impact and bypass characterizing the posterior. Rosenberg et al. (2007) presents a model where agents can only infer others' private signals from their actions. They show that an active player should be interpreted as a positive signal. This has a similar flavor as my result (Proposition 2). Smith et al. (2017) point out that more optimistic priors do not necessarily lead to more optimistic posteriors. This non-monotonicity is the core analysis in many of my results. I contribute to this branch of literature by setting up a framework to systematically study these effects.

This paper proceeds as follows. Section 2 sets up the model and characterizes the posteriors. Section 3 presents the main results, comparing the sophisticated and the naive posterior, and providing comparative statics. Section 4 revisits the applications and talks about how my results imply about them. Section 5 summarizes, discussing possible sequels and extensions. All proofs are in the Appendix.

2 Model

Outside observers (we) are interested in the value of a random variable P with realizations $p \in [0,1]$. Our prior about P has a continuous density $\pi(p)$. We don't have direct access to any signals. Instead, we learn from an insider (he) who faces an optimal stopping problem where the stopping rule is determined by his signals about P. I first describe the environment faced by both parties.

A. The experimenter's problem

Starting from a publicly known period $t = -T$, the insider undertakes an experiment generating an i.i.d signal $S_t \in \{0,1\}$ in each period $t \in \{-T,\ldots,-1,0,1,\ldots\}$, where T is a positive integer and $\Pr(S_t = 1) = p$. I call the signal realization a success if $s_t = 1$ and a failure if $s_t = 0$. As a lead example, consider S_t to be the reward of a binary bandit that yields 1 with chance P and 0 with chance $1-P$. The insider is Bayesian rational and updates his belief after each signal realization, based on which, he decides whether to stop experimenting.

Call the sequence of signal realizations up to period $z \geq -T$ the *full history* at z and denote it by $\mathcal{H}^z = \langle s_t \rangle_{t=-T}^{z}$, or $\langle s_{-T},\ldots,s_z \rangle$. A full history belongs to the set of all binary sequences

$$\mathcal{S} = \{\langle s_t \rangle_{t=-T}^{z} | s_t \in \{0,1\},\ -T \leq z\}\ .$$

After seeing the full history $\mathcal{H}^z$ in each period z, assume the insider decides whether to stop according to a stopping strategy determined by an *index* $I(\cdot) : \mathcal{S} \to \mathbb{R}$. Specifically, I call a full history $\mathcal{H}^z$ *admissible* if $I(\mathcal{H}^z) \geq 0$. The insider continues[4] in period z if and only if the full history $\langle s_t \rangle_{t=-T}^{z}$ is admissible throughout its path, i.e. $I(\langle s_t \rangle_{t=-T}^{\tau}) \geq 0$ for $\tau = -T,\ldots,z-1$. Indexes that are equal under a monotone transformation that preserves the sign are considered equivalent.

I focus on the case where the insider continues only if the full history is sufficiently encouraging, in the sense that the index $I(\cdot)$ weakly rises (resp. falls) in the number of successes (resp. failures) in the full history. Let $\sigma(\langle s_t \rangle_{t=-T}^{z}) = \sum_{t=-T}^{z} s_t$ and $\varphi(\langle s_t \rangle_{t=-T}^{z}) = \sum_{t=-T}^{z}(1-s_t)$ count the number of 1s and 0s in an arbitrary binary sequence $\langle s_t \rangle_{t=-T}^{z} \in \mathcal{S}$. Assume that $I(\mathcal{H}) \geq I(\mathcal{H}')$ if $\sigma(\mathcal{H}) \geq \sigma(\mathcal{H}')$ and $\varphi(\mathcal{H}) \leq \varphi(\mathcal{H}')$.

In the one-armed bandit problem, suppose the insider is risk-neutral and has a uniform prior on P. He obtains the random reward S_t but pays a fixed cost c for each trial. He stops experimenting if he is sufficiently convinced that the expectation of P is less than c. The index $I(\cdot)$ equals to the Gittins index[5] minus c, which is composed of the running payoff expectation, namely $\sigma(\cdot)/(\sigma(\cdot)+\varphi(\cdot))-c$, and a term representing the information value of each trial. For another instance, consider the gambler's ruin

[4] Note that the insider always continues when indifferent.

[5] There is no closed-form expression of the Gittins index in this context as it is recursively computed. It satisfies the required property of rising in $\sigma(\cdot)$ and falling in $\varphi(\cdot)$, see Banks and Sundaram (1992).

problem[6], the index is the insider's running wealth level. Specifically, if he starts with wealth 2, wins 1 with each success and loses 1 with each failure, and continues until he goes broke. The index $I(\cdot) = 2 + \sigma(\cdot) - \varphi(\cdot)$.

B. The observer's problem

We arrive at period 0 and begin observation till period $Z \geq 0$ while the experiment is ongoing. We see the *public history* $H = \langle s_t \rangle_{t=0}^{Z}$, but not the *prehistory* $H^- = \langle s_t \rangle_{t=-T}^{-1}$. Recall that the insider sees the full history[7] $\mathcal{H}^Z = H^- + H$ at Z. We can also interpret the experiment starting time T as the *prehistory length*. Assume the insider's stopping strategy I is publicly known so that we as the outside observer treat I as exogenously given. The objective is to learn the underlying payoff P given the above information.

To picture this scenario, let's place our gambler in a casino. We come to watch him playing for a while, seeing a sequence of rewards, which is the public history H. To estimate the payoff probability P, I first consider as a benchmark the *naive posterior* based solely on H. Let $G_p(H)$ denote its distribution. The naive posterior is derived from the basic Bayes' rule,

$$G_p(H) = \frac{\int_0^p q^{\sigma(H)}(1-q)^{\varphi(H)}\pi(q)\mathrm{d}q}{\int_0^1 q^{\sigma(H)}(1-q)^{\varphi(H)}\pi(q)\mathrm{d}q}\,. \tag{1}$$

On the other hand, we know that the insider had played T rounds before we arrive, and that he would have left if the unobserved rewards, namely the prehistory H^- were poor. In contrast to the naive posterior $G_p(H)$, we compose the *sophisticated posterior* $F_p(H; I, T)$ that accounts for all possible prehistories that are consistent with our observation — firstly, the fact that the experiment is still ongoing; secondly, the public history H.

To be consistent with the first part, the prehistories considered in the sophisticated posterior must be admissible. Therefore I first characterize the set of all unconditional admissible prehistories:

$$\Gamma_{\varnothing}(I, T) = \left\{ \langle s_t \rangle_{t=-T}^{-1} \in \mathcal{S} | I(\langle s_t \rangle_{t=-T}^{z}) \geq 0,\ \text{for all } z = -T, \ldots, -1 \right\},$$

where the empty set sign in the subscript refers to the its unconditional nature. This is the set of all full histories that pass the cut-off up to period -1. But this does not conclude our inference. The information contained in the public history H again refines the set of prehistories that we need to consider. Here I give an example.

[6]The gambler's ruin problem per se is not an optimal stopping problem, but it is easy to formulate stopping problems that apply this index. Say the insider enjoys high utility from experimenting regardless of P, but is constrained by his budget.

[7]"$+$" between sequences denotes the concatenation operation. For instance, $\langle 10 \rangle + \langle 01 \rangle = \langle 1001 \rangle$.

Example 1. Let the index $I(\cdot) = \sigma(\cdot) - \varphi(\cdot)$ and assume prehistory length $T = 2$. The admissible prehistories are thus $\Gamma_{\varnothing}(I, T) = \{\langle 10\rangle, \langle 11\rangle\}$. However, if we see public history $H = \langle 00\rangle$, the prehistory $\langle 10\rangle$ is not possible because the insider would have stopped in period 1.

This example shows that our observation shapes our inference about the unobserved signal realizations. Hence, we need to further narrow down to the prehistories that are consistent with the specific public history H. I say a prehistory is *H-admissible* if it is still admissible when followed by the given public history H. Let $\Gamma_H(I, T)$ be the *refined prehistory* set, namely set of all H-admissible prehistories, which are the possible prehistories conditional on observing H,

$$\Gamma_H(I, T) = \{H^- \in \Gamma_{\varnothing}(I, T) | H^- + H \in \Gamma_{\varnothing}(I, T + Z + 1)\} .$$

This prunes our inferred prehistories as $\Gamma_H \subseteq \Gamma_{\varnothing}$ for any $H \in \mathcal{S}$. Now I formally express $F(H; I, T)$ as an average on all possible paths of the full history,

$$F_p(H; I, T) = \frac{\int_0^p \left(\sum_{H^{-1} \in \Gamma_H(I,T)} q^{\sigma(H^{-1}+H)} (1-q)^{\varphi(H^{-1}+H)}\right) \pi(q) \mathrm{d}q}{\int_0^1 \left(\sum_{H^{-1} \in \Gamma_H(I,T)} q^{\sigma(H^{-1}+H)} (1-q)^{\varphi(H^{-1}+H)}\right) \pi(q) \mathrm{d}q} . \quad (2)$$

The difference between the sophisticated posterior F and the naive posterior G is essentially the inference about the unobserved prehistory, reflected in the summation inside the integration in (2). I call it the *proportional density* of H. With some rearrangements, I denote it as

$$f_p(H; I, T) = \left(\sum_{H^- \in \Gamma_H(I,T)} p^{\sigma(H^-)} (1-p)^{\varphi(H^-)}\right) p^{\sigma(H)} (1-p)^{\varphi(H)} . \quad (3)$$

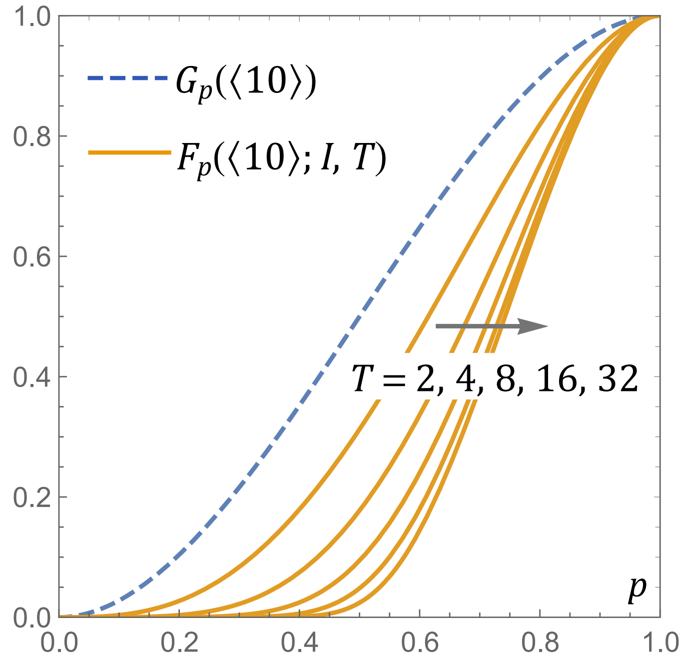
Generally, the sophisticated posterior is very hard to compute, even for extremely simple forms of I. For instance, let's apply the specifications in example 1, where the insider continues the experiment if the number of successes is greater than that of failures along the path, namely $I(\cdot) = \sigma(\cdot) - \varphi(\cdot)$. The computation of the sophisticated posterior involves combinatorial numbers:

$$F_p(H; I, T) = \frac{\int_0^p \sum_{m=0}^{\lceil T/2 \rceil} c(m, 2, T+1-2m) q^{T-m} (1-q)^m \mathrm{d}q}{\int_0^1 \sum_{m=0}^{\lceil T/2 \rceil} c(m, 2, T+1-2m) q^{T-m} (1-q)^m \mathrm{d}q} , \quad (4)$$

where [8] $c(m, \rho, r) = \frac{r}{m\rho + r} \binom{m\rho + r}{m}$ and $\lceil \cdot \rceil$ denotes the ceiling operation.

[8] It is the fuss-catalan number (Aval (2008)) that counts the number of quasi staircase walks from two nodes on a grid. Let the full histories travel on a grid where each success (failure) leads to a vertical (horizontal) movement, the paths are quasi staircase walks.

Figure 1: Comparison between the naive and the sophisticated posterior



The index $I(\cdot) = \sigma(\cdot) - \varphi(\cdot)$, prehistory length $T = 2$, and public history $H = \langle 10 \rangle$. The distance between the naive and sophisticated posteriors could very large.

To show why it is worth the effort to compute the sophisticated posterior, let's plug in $H = \langle 10 \rangle$ and $T = 2$. Assuming uniform prior, namely $\pi(p) = 1$ for $p \in [0, 1]$, I plot the resulting $G_p(H)$ and $F_p(H; I, T)$ in Figure 1. The difference between the naive and sophisticated posterior could be considerably large. The expectation of P is 0.5 based on G, but 0.6 based on F, a 20% difference. Besides, the index $I(\cdot) = \sigma(\cdot) - \varphi(\cdot)$ implies that there cannot be more failures than successes at any point in the prehistory. As T grows large, the sophisticated posterior puts probability approaching[9] 1 on $P \geq 0.5$, while the naive posterior always put higher possibility on $P < 0.5$ as long as $\sigma(H) < \varphi(H)$.

In the next section, I show that the bias shown in the above example is persistent and dynamic. I also provide closer comparison between the naive and the sophisticated posterior and show that many seemingly obvious comparative statics for the naive posteriordo not hold for the sophisticated posterior.

[9] It is similar to the Ballot problem (`https://mathworld.wolfram.com/BallotProblem.html`). If the realization of P is p, the probability that there are no more failures than successes along the history is $\sum_{m=0}^{\lceil T/2 \rceil} c(m, 2, T{+}1{-}2m) p^{T-m} (1-p)^m$ where $c(m, \rho, r)$ is as defined in (4). As $T \to \infty$, this expression appoaches $2(p - 0.5)/p$ if $p > 0.5$ and 0 if $p \leq 0.5$. Thus $\lim_{T \to \infty} F_p(H, I, T) = 0$ for $p < 0.5$. The generalized result is in the Appendix

3 Main Results

First, I show that the naive posterior is equivalent to the sophisticated posterior if the insider never stops, namely the index is always nonnegative. In this case, knowing the existence of prehistories does not convey any information about the underlying random payoff. The refined prehistory set Γ_H is simply all length T binary sequences. The first term in the proportional density (3) simplifies to

$$\sum_{H^- \in \Gamma_H(I_0,T)} q^{\sigma(H^-)}(1-q)^{\varphi(H^-)} = \sum_{k=0}^{T} \binom{T}{k} q^k (1-q)^{T-k} \equiv 1 \, .$$

Therefore, the expression of the naive posterior (1) is equivalent to that of the sophisticated posterior (2). If the insider uses a general index that does filter out some prehistories, the coefficient of $q^{\sigma(H^-)}(1-q)^{\varphi(H^-)}$ with small $\sigma(H^-)$ becomes 0, thus increasing the sophisticated posterior. The comparison between the sophisticated and the naive posterior can be generalized to the comparative statics of the sophisticated posterior given different stopping strategies.

I say an index I_2 is (weakly) *harsher* than index I_1 if $I_2(\cdot) \geq 0$ implies $I_1(\cdot) \geq 0$, namely $\{\mathcal{H} \in \mathcal{S} | I_2(\mathcal{H}) \geq 0\} \subseteq \{\mathcal{H} \in \mathcal{S} | I_1(\mathcal{H}) \geq 0\}$. This implies that $\Gamma_H(I_1, T) \subseteq \Gamma_H(I_2, T)$. For instance, I_2 is harsher than index I_1 if $I_1(\mathcal{H}) \geq I_2(\mathcal{H})$ for all $\mathcal{H} \in \mathcal{S}$. A harsher index implies a stricter stopping strategy, namely the insider is more likely to stop. An index I is least harsh if $\Gamma_H(I_0, T) = \{\langle s_t \rangle_{t=-T}^{-1} | s_t \in \{0,1\}\}$, namely $I(\mathcal{H}) \geq 0$ for all $\mathcal{H}$. I use $I_0(\cdot) \equiv 1$ to represent a least harsh index.

I employ the *likelihood ratio order* $\succeq_{\text{lr}}$ for the comparison of distributions. Note that the sophisticated posterior is differentiable as I assumed continuous prior. For two arbitrary differentiable distributions F_p^1 and F_p^2, I say F^1 dominates F^2 in the likelihood ratio order[10] if the ratio of their density functions, $(\partial F_p^1/\partial p)/(\partial F_p^2/\partial p)$, rises in $p \in (0,1)$. I denote this relation as denoted as $F^1 \succeq_{\text{lr}} F^2$. I first show that a harsher index leads to a stochastically higher sophisticated posterior.

Lemma 1. *For all public histories, the sophisticated posterior rises in the likelihood ratio order when the index is harsher , i.e. for $\forall H$ and T,*

$$\Gamma_H(I_1, T) \subseteq \Gamma_H(I_2, T) \Rightarrow F(H; I_2, T) \succeq_{lr} F(H; I_1, T) \, .$$

The index function weakly increases with the number of successes and falls in the number of failures. A harsher index filters out the worse prehistories that contain more failures and fewer successes. The first term in the proportional density (3) thus contains more terms with high powers of p and low powers of $1-p$, resulting in a

[10] Note that the likelihood ratio order is a strong stochastic ordering. It implies the first order stochastic dominance and the relationship preserves for different priors.

stochastically higher posterior in the sense of likelihood ratio order.

Since the naive posterior is equivalent to the sophisticated posterior with the least harsh index I_0. It follows easily that the sophisticated posterior dominates the naive posterior in the likelihood ratio order for all public histories, i.e, for all H and T,

$$F(H; I, T) \succeq_{\text{lr}} F(H; I_0, T) \equiv G(H) \; . \tag{5}$$

In other words, the naive posterior is always downwardly biased. Next, I present a more specific comparison between the naive and the sophisticated posterior, which reflects the dynamic features of this bias.

First consider the linear index $I(\cdot) = \sigma(\cdot) - a\varphi(\cdot) + c$ where $a, c \geq 0$. It applies to the gambler's ruin like problems and is related to a large family of indexes. For instance in the bandit problem, the Gittins index equals the expectation of P, namely $\sigma(\mathcal{H})/(\sigma(\mathcal{H}) + \varphi(\mathcal{H}))$ if the insider is sufficiently impatient[11], or approaches it as the full history $\mathcal{H}$ gets long. In these cases if the experimenting cost is $a/(a+1)$, the index of the stopping rule is equivalent to $I(\cdot) = \sigma(\cdot) - a\varphi(\cdot)$. For any cost in $(0.5, 1)$, we can find a linear I with sufficiently large c such that the index in the bandit problem is harsher than it.

Next, I discuss the special case where $I(\cdot)$ is harsher than $\sigma(\cdot) - a\varphi(\cdot)$ where a is an integer. It is easy to infer that the first a signal realizations must be 1 if $T \geq a$, namely $\langle s_t \rangle_{t=-T}^{-T+a-1}$ are all 1's, otherwise the insider would have stopped. This is as if we observe a more successes. Using the regular expression, where $\langle 1\{n\}0\{m\}\rangle$ stands for a sequence of n 1's followed by m 0's, we have $F(H; I, T) \succeq_{\text{lr}} G(H + \langle 1\{a\}\rangle)$. Apart from this observation and (5), I present the following general result.

Proposition 1. *If the index $I(\cdot)$ is harsher than $\sigma(\cdot) - a\varphi(\cdot) + c$ and $T \geq (2a+1)^2 - 2$, the sophisticated posterior is higher than the naive posterior with an expanded public history, specifically,*

$$F(H; I, T) \succeq_{lr} G(\tilde{H} + H)$$

where (a) $\tilde{H} = \langle 1\{4\lfloor a\rfloor - \lceil c\rceil\}0\rangle$ and (b) $\tilde{H} = \langle 1\{4\lfloor a\rfloor\}0\{\lceil c/a\rceil + 1\}\rangle$.

When the original public history H contain more failures than successes, the expanded public history is more favorable than the original one and thus this comparison is stronger than that $F(H; I, T) \succeq_{\text{lr}} G(H)$. This result also provides a useful rule of thumb if the stopping strategy is harsh, say $a \geq 1$ and $0 \leq c < a$. For instance, suppose $I(\cdot) = \sigma(\cdot) - \phi(\cdot)$, namely the insider stops as long as there are more failures than successes. The prehistory is likely more favorable than $\langle 11110\rangle$ if he has continued for more than 7 periods. I provide an even tighter bound and discuss the asymptotic property of the sophisticated posterior in the appendix.

[11] See Fristedt and Berry (1988)

To give an intuition, consider the two extreme cases in the above example, that the prehistory contains only successes, or half-successes-half-failures. An averaged guess is that there were twice as many successes as failures[12]. But since the sophisticated posterior puts high probability on $P \geq 0.5$, prehistories with more successes gain higher likelihood. It is most likely that the prehistory contain more than twice as many successes than failures. But it is hard to compare $F(H; I, T)$ and $G(H^- + H)$ with any certain H^- since F is more diverse. To account for the uncertainty, we only guess that part of the prehistory is more favorable than $\langle 11110 \rangle$.

Proposition 1 also implies that the sophisticated posterior $F(H; I, T)$ critically depends on the prehistory length T. Note that Lemma 1 corresponds to the intuition that F rises if the insider is more prone to stop. A similar conjecture is that F would rise when the insider has made more stopping decisions, namely $F(H; I, T)$ rises in T, as reflected in Figure 1. Imagine we observe a gambler playing a bandit, it is tempting to guess that the bandit yields rewards more frequently if it had kept the gambler playing for 20 rounds than if it is only played for 10 rounds.

Whether the above intuition is correct critically depends on the property of the public history that we observe. It does not hold if the public history contains many failures. For instance, if we see 3 consecutive failures and there was only one unobserved signal before them, we are pretty sure it was a success, as it is unlikely that the insider would keep going after a full history of entire failures. On the other hand, if there were 10 previous signals, this assurance is lost. Next, I show a detailed example.

Example 2. Let the index $I(\cdot) = \sigma(\cdot) - \varphi(\cdot) + 1$ and public history $H = \langle 01 \rangle$. First consider the prehistory length $T_1 = 1$; the only refined prehistory is $\langle 1 \rangle$. Thus the proportional density $f_p(H; I, T_1) = p^2(1-p)$. Next, let $T_2 = 2$; the refined prehistories becomes $\langle 11 \rangle$, $\langle 10 \rangle$, and $\langle 01 \rangle$, thus $f_p(H; I, T_2) = p^3(1-p) + 2p^2(1-p)^2$. Note that the likelihood ratio $f_p(H; I, T_1)/f_p(H; I, T_2) = 1/(2-p)$ rises in p, therefore $F(H; I, T_1) \succeq_{\text{lr}} F(H; I, T_2)$. Thus the sophisticated posterior could fall in the prehistory length.

By this example, we see that combining the possibilities of a prehistory of two successes and another of one-success-one-failure leads to worse estimation than a certain prehistory that contains a single success. The addtional signal that comes with the longer prehistory can be either a success or a failure. Which case is more likely depends on the properties of H and T. Example 1 showed that some public histories further refine the prehistory set, since they exclude the prehistories after which the insider would stop experimenting after seeing (part of) H. According to whether a public history may induce stopping after certain prehistories, I classify it into the following categories.

[12] By (4), there are $c(m, \rho, T+1-2m)$ admissible prehistories with $T-m$ successes and m failures. Thus there are most different pathes for prehistories with $\lfloor 2T/3 \rfloor$ successes and $\lceil T/3 \rceil$ failures. So this is indeed a good guess if the success chance $P = 0.5$.

Definition. A public history H is *decisive* (resp. *indecisive*) for index I and prehistory length T if $\Gamma_H(I,T) \subset \Gamma_\varnothing(I,T)$ (resp. $\Gamma_H(I,T) = \Gamma_\varnothing(I,T)$).

An indecisive public history does not screen out any admissible prehistories, while for a decisive public history, the refined prehistory set is a real subset of the admissible prehistory set. For example, let $I(\cdot) = \sigma(\cdot) - \varphi(\cdot)$, $T = 2$, and public history length $Z = 1$. The admissible prehistories are $\langle 11 \rangle$ and $\langle 10 \rangle$. The public history $\langle 0 \rangle$ is decisive since the only $\langle 0 \rangle$-admissible prehistory is $\langle 11 \rangle$. On the other hand, $\langle 1 \rangle$ is indecisive as all admissible prehistories are $\langle 1 \rangle$-admissible.

Since $I(\cdot)$ rises in $\sigma(\cdot)$, any public history that contains only successes is indecisive. On the other hand, a decisive public history with fixed length Z does not always exist. For instance there is no decisive public history with $Z = 2$ if T is odd in the above example. If Z is flexible, indecisive public histories that contain failures exist as long as the marginal effect of successes and failures on I are bounded away from 0 and $-\infty$ respectively. Specifically, if $I(\mathcal{H}+\langle 1 \rangle) - I(\mathcal{H}) \geq m > 0$ and $I(\mathcal{H}) - I(\mathcal{H}+\langle 0 \rangle) \leq M < \infty$ for all $\mathcal{H}$, the public history $\langle 1\{a\}0\{b\} \rangle$ where $a/b \geq \lceil M/m \rceil$ is indecisive at all T. Indecisive public histories usually have many early successes.

Definition. The prehistory length T is *critical* (resp. *noncritical*) for public history H if $\Gamma_{\langle 0 \rangle + H}(I,T) \subset \Gamma_H(I,T)$ (resp. $\Gamma_{\langle 0 \rangle + H}(I,T) = \Gamma_H(I,T)$).

At a critical T, the refined admissible prehistory shrinks when H is prepended with a failure, while at a critical T, the refined admissible prehistory does not change. Continuing the previous example, T is noncritical for $\langle 0 \rangle$ as $\Gamma_{\langle 0 \rangle} = \Gamma_{\langle 00 \rangle} = \{\langle 11 \rangle\}$, while critical for $\langle 1 \rangle$ since $\Gamma_{\langle 1 \rangle} = \{\langle 11 \rangle, \langle 10 \rangle\}$ and $\Gamma_{\langle 01 \rangle} = \{\langle 11 \rangle\}$. Note that at some critical T, it is not possible to observe $\langle 0 \rangle + H$, in which case $\Gamma_{\langle 0 \rangle + H} = \emptyset \subset \Gamma_H$. For instance T is critical for $\langle 00 \rangle$ as $\Gamma_{\langle 000 \rangle} = \emptyset$. Combining the properties of T and H characterizes how the prehistory set changes in time, leading to the following lemma.

Lemma 2. *(a) If T is critical for H, the sophisticated posterior locally rises in T, in the sense that $F(H; I, T+1) \succeq_{lr} F(H; I, T)$;*

(b) If T is noncritical for a indecisive public history H, the sophisticated posterior does not change in T, i.e., $F(H; I, T+1) \equiv F(H; I, T)$;

(c) If T is noncritical for a decisive public history H, the sophisticated posterior locally falls in T, in the sense that $F(H; I, T) \succeq_{lr} F(H; I, T+1)$.

To outline the proof, first consider case (b). If H is noncritical, $H^- + \langle 0 \rangle \in \Gamma_H(I, T+1)$ for any $H^- \in \Gamma_H(I,T)$. If H is also indecisive, we have $\Gamma_H(I,t) = \Gamma_\varnothing(I,t)$ for both $t = T, T+1$. The sets of H-admissible prehistories of length T and $T+1$ satisfy the following relation,

$$\Gamma_H(I, T+1) = \{H^- + \langle 0 \rangle, H^- + \langle 1 \rangle | H^- \in \Gamma_H(I,T)\} \,. \tag{6}$$

In other words, any H-admissible histories in T is still admissible in $T+1$ if followed by a failure. The additional signal realization that comes with the longer prehistory does not carry any new information. Thus the sophisticated posterior does not change, namely $F(H; I, T+1) \equiv F(H; I, T)$. In case (c) where H is decisive, the evolution from $\Gamma_H(I, T)$ to $\Gamma_H(I, T+1)$ does not obey the equivalence (6). Recall example 2, one of the H-admissible prehistory in $T+1$, $\langle 01 \rangle$, cannot be derived from any H-admissible prehistory in T by appending a new signal realization. In this example,

$$\Gamma_H(I, T+1) = \{H^- + \langle 0 \rangle, H^- + \langle 1 \rangle | H^- \in \Gamma_H(I, T)\} \cup \{\langle 01 \rangle\} .$$

There are more ways to reach the state of one more failure through the prehistory $\langle 01 \rangle$, thus the sophisticated posterior falls.

In case (a), since $\Gamma_{\langle 0 \rangle + H}(I, T) \subset \Gamma_H(I, T)$, the H-admissible prehistories at T with the least successes and most failures are not admissble at $T+1$ if followed by a failure, thus

$$\Gamma_H(I, T+1) = \left\{ H^- + \langle 0 \rangle, H^- + \langle 1 \rangle | H^- \in \Gamma_H(I, T) \backslash \{ \arg\max_{H^- \in \Gamma_H(I,T)} \varphi(H^-) \} \right\} . \tag{7}$$

Compared with (6), the H-admissible prehistory is more likely to contain more successes, thus $F(H; I, T+1) \succeq_{\text{lr}} F(H; I, T)$.

Note that it is not possible that a public history is alway noncritical as T grows unless the insider never stops, where the sophisticated posterior is equivalent to the naive one. On the other hand, it is possible that a public history is indecisive for any T, as mentioned after Example **??**. The result in Proposition 2(a) thus rolls over as T continues to rise and that the sophisticated posterior rises in T could be extended to a global property. Note that in Figure 1, the public history $\langle 10 \rangle$ is always indecisive for $I(\cdot) = \sigma(\cdot) - \varphi(\cdot)$, and thus $F(H; I, T)$ always rises in T.

Proposition 2. *If the public history is indecisive at any T, longer prehistory length implies a higher sophisticated posterior in the likelihood ratio order, i.e.,*

$$T' \geq T \Rightarrow F(H; I, T') \succeq_{lr} F(H; I, T) .$$

Observing an indecisive public history does not screen out any prehistories, thus the prehistory length T plays the central role in determining the sophisticated posterior. With this family of public histories, the sophisticated posterior stochastically rises with longer prehistory, growing further away from the naive posterior. In other words, the bias accumulates as the prehistory length rises.

Note that the empty public history $H = \varnothing$ is always indecisive. Corollary 2 implies that the sheer information that the experiment has continued for T period reveals much about the underlying random payoff. The experiment that survives a longer time implies a stochastically higher underlying payoff. Turn to the gambler example, even if we cannot see the rewards, we can better evaluate the yield probabilities of the bandit just by observing the experimenting time.

Another aspect of the dynamic feature of the bias is its path-dependent property. I demonstrate this by showing how the sophisticated posterior changes when the order of successes and failures in the public history is altered. To capture this ordering, note that if $H' = \langle s'_t \rangle_{t=0}^{Z}$ has earlier failures than $H = \langle s_t \rangle_{t=0}^{Z}$, the running summation of H' is always lower than that of H, namely $\sigma(\langle s'_t \rangle_{t=0}^{z}) \leq \sigma(\langle s_t \rangle_{t=0}^{z})$ for $z = 0, \ldots, Z$. For instance, let $H' = \langle 011 \rangle$ and $H = \langle 101 \rangle$. Their running summations are respectively $0, 1, 2$ and $1, 1, 2$. With this characterization, I find that the sophisticated posterior stochastically rises when failures come earlier.

Proposition 3. *The sophisticated posterior is higher in the likelihood ratio order if it has earlier failures, i.e. for* $H = \langle s_t \rangle_{t=0}^{Z}$ *and* $H' = \langle s'_t \rangle_{t=0}^{Z}$ *where* $\sigma(H') = \sigma(H)$,

$$\sigma(\langle s'_t \rangle_{t=0}^{z}) \leq \sigma(\langle s_t \rangle_{t=0}^{z}) \text{ for } z = 0, \ldots, Z \Rightarrow F(H'; I, T) \succeq_{lr} F(H; I, T) \,.$$

A public history with early failures screens out more bad prehistories, as it is less likely that the insider would push forward after a history that continues to worsen. Specifically, if H and H' differ only in that H' has earlier failures, we have $\Gamma_{H'}(I,T) \subseteq \Gamma_H(I,T)$. By similar reasoning of Lemma 1, the sophisticated posterior is driven up. Note that if both H and H' are indecisive, this effect does not exist as their refind prehistory sets are both the admissible prehistory set $\Gamma_\varnothing$, thus $F(H'; I, T) \equiv F(H; I, T)$.

This result also shows that, to compute the sophisticated posterior, the number of successes and failures are no longer sufficient statistics as in deriving the naive posterior. It it important to keep track of the ordering of different signal realizations in the public history, whose effect can sometimes outweigh their numbers. In other words, a public history with more successes may lead to lower estimation than a public history with less but later succeses. Here I present an example.

Example 3. Let the prehistory length $T = 2$, and index $I(\cdot) = \sigma(\cdot) - 2\varphi(\cdot) + 2$. Consider public histories $H_1 = \langle 01 \rangle$ and $H_2 = \langle 101 \rangle$. The only H_1-admissible prehistory is $\langle 11 \rangle$, and the H_2-admissible prehistories are $\langle 11 \rangle$, $\langle 10 \rangle$, and $\langle 01 \rangle$. Hence, the proportional densities $f_p(H_1) = p^3(1-p)$ and $f_p(H_2) = p^4(1-p) + 2p^3(1-p)^2$. The likelihood ratio is thus $f_p(H_1)/f_p(H_2) = 1/(2-p)$, which rises in p. Thus $F(H_1; I, T) \succeq_{\text{lr}} F(H_2; I, T)$, though H_2 contain more successes.

Having a later failure allows for prehistories $\langle 10 \rangle$ and $\langle 01 \rangle$, significantly expanding the refined prehistory set. This effect can dominate that of the additional success and lowers the sophisticated postrior. An important message from the above analysis is that, the monotonicity of the naive posterior in the number of successes and failures does not carry through to the sophisticated posterior.

Proposition 3 shows that the sophisticated posterior sensitively depends on the pattern of the public history. Now I study how it changes when we marginally alter the public history while maintaining its pattern. Specifically, what's the effect of one additional observed signal realization. If the new signal comes after the original public history, it is as if we continue the observation for one period and see a subsequent signal realization. Now the public history H becomes $H + \langle s_{Z+1}\rangle$, where $s_{Z+1} \in \{0, 1\}$; If the new signal comes before the original public history, it is as uncovering one unobserved signal in the prehistory.

It is trivially true that subsequent successes always stochastically increases the sophisticated posterior, namely $F(H + \langle 1\rangle; I, T) \succeq_{\text{lr}} F(H; I, T)$. The following success does not affect our inference of the prehistory, hence $f_p(H + \langle 1\rangle; I, T) = p \cdot f_p(H; I, T)$. The interesting case is that, it is not necessarily true that a subsequent failure will reduce the sophisticated posterior. The experiment could have stopped after our original observation periods, and we wouldn't know it until we continue to the next period. If we do, the information that the experiment is still going on may compensate for the negative effect of an additional failure. Here's an example.

Example 4. Let the index $I(\cdot) = \sigma(\cdot) - \varphi(\cdot)$, prehistory length $T = 2$, and public histories $H_1 = \langle 0\rangle$ and $H_2 = \langle 00\rangle$. The H_1-admissible prehistories are $\langle 11\rangle$, $\langle 10\rangle$ and $\langle 01\rangle$, while the only H_2-admissible prehistory is $\langle 1\rangle$. Now, $f_p(H_1; I, 1) = p^2(1-p)+2p(1-p)^2$ and $f_p(H_2; I, 2) = p^2(1-p)^2$. Given certain prior of P, their relation can be ranked in the second order stochastic dominance [13] $\succeq_2$. For instance, when the prior $\pi(p) = 1$, we have $F(H_2; I, T) \succeq_2 F(H_1; I, T)$, though H_2 contain more failures.

Similar to Example 3, which showed that more successes does not always increase our estimation, we see that many seemingly obvious comparative statics results for the naive posteriors do not hold for the sophisticated posterior. Another message from this finding is that more information could lead to a larger bias. While additional failures always decrease the naive posterior, it may increase the sophisticated posterior. A sufficient condition to make sure that observing another failure that follows H reduces the sophisticated posterior is that H is noncritical.

On the other hand, uncovering more signal realizations in the prehistory always leads to the usual comparative statics results. Specifically, if the signal realization in period $-t$ for some $1 \le t \le T$ is revealed, knowing $S_{-t} = 1$ increases the sophisticated posterior and knowing $S_{-t} = 0$ lowers the sophisticated posterior. Moreover, the position of the revealed signal, $-t$, does not matter. In other words, uncovering a signal in either the recent or the old prehistory has the same effect on the sophisticated posterior. Denote the sophisticated posterior with the additional information that $S_{-t} = i$ as $F(H; I, T | S_{-t} = i)$ for $i \in \{0, 1\}$, we have the following result.

[13] For arbitrary distributions F and G, I say $F \succeq_2 G$ if $\int_0^q (F_p - G_p)\,\mathrm{d}p \le 0$ for $q \in [0, 1]$ In Example 4, we have $F_p(\langle 0\rangle; I, T) = p^4 - 4p^3 + 4p^2$ and $F_p(\langle 00\rangle; I, T) = 6p^5 - 15p^4 + 10p^3$. Thus $\int_0^q (F_p(H_2; I, T) - F_p(H_1; I, T))\,\mathrm{d}p = q^6 - 16q^5/5 + 7q^4/2 - 4q/3 < 0$ when $q \in [0, 1]$. Note that unlike having more precedent successes as in Example 3, subsequent failures do not lead to shifts in the likelihood ratio order. This is because a following failure excludes at most the worst prehistory, whose effect is less dominating.

Proposition 4. *(a) Observing a previous success (failure) increases (lowers) the sophisticated posterior in the likelihood ratio order, i.e. for $T \geq 1$ and $1 \leq t \leq T$,*

$$F(H; I, T | S_{-t} = 1) \succeq_{lr} F(H; I, T) \succeq_{lr} F(H; I, T | S_{-t} = -1) \, ;$$

(b) For any $1 \leq t, t' \leq T$ and $s \in \{0, 1\}$,

$$F(H; I, T | S_{-t} = s) = F(H; I, T | S_{-t'} = s) \, .$$

Note that if $t = 1$, namely the most recent signal in the prehistory is uncovered and become part of the public history, Proposition 4(a) implies

$$F(\langle 1 \rangle + H; I, T-1) \succeq_{\mathrm{lr}} F(H; I, T) \succeq_{\mathrm{lr}} F(\langle 0 \rangle + H; I, T-1) \, ,$$

such that the marginal effect of a previous success (failure) is always positive (negative). Discovering additional successes (failures) in the prehistory not only excludes the prehistories that have more failures (successes), but also increases the relative weights of good (bad) prehistories. While uncovering more unobserved signals refines the admissible prehistory set, the position of whom does not matter.

4 Application

To illustrate how the previous analysis implies about real-world scenarios, in this section I revisit some of the applications mentioned in Section 1.

Investing in new mutual funds We can observe the daily yields of a relatively new mutual fund, but there is a prehistory since the fund manager (he) must have tracked the performance of the portfolio before launching the product. He may also adjust the asset allocation if it is sufficiently poor. Both decisions can be embedded in the index rule. The evaluation using only the observed yield data is downwardly biased. This helps to justify the lock-up periods as investors may redeem the investment too soon due to undervaluation. By Lemma 1 and Proportion 2, this bias is stronger if the manager is more selective or more patient, namely, he has a harsh stopping strategy, or he is prudent in launching a product. By Proposition 3, observing three days of falls and then three days of rises is more favorable information than the reverse case.

IPO underpricing Asymmetric information leads to unintentional IPO underpricing, as young firms lack historical financial data and underwriters do not fully account for their inexplicit gains. Tech firms usually experience a high level of underpricing (see Berggren (2017)), which fact is inline with the implication of this paper since they have a long prehistory of R&D input. China also sees extreme IPO underpricing (see Chan et al. (2004)), which can partly be attributed to the institutional factors such that Chinese firms survived harsher standards to go on public.

College admission Making admission decisions only on the score of one exam mimics the naive posterior that ignores the prehistory of the students' previous performance. The final score may either overestimate or underestimate a student's ability. While students from wealthy families suffer more symmetric randomness, those with financial constraints, are more prone to drop out early. This policy may lead to inequality against students from low-income families or certain ethnic groups in some countries, who face harsher self-selection.

5 Summary and Discussion

This paper studies an observational learning problem where the noise in the Bayesian inference owes to an unobserved prehistory motivating the observed party. I show that the naive posterior based solely on the observed public history is biased. I characterize the sophisticated posterior that accounts for all possible prehistories consistent with the observed history and the stopping problem. Many seemingly obvious comparative statics results for the naive posteriors do not carry through to the sophisticated posterior. The path-dependent property and other unusual comparative statics results of the sophisticated posterior again show the importance of making comprehensive inferences from the prehistory, rendering the naive posterior inappropriate.

I focus on the case where the stopping rule and experimenting time are known. Though my results do not rely on exact knowledge of those premises, the case of observing stopping is trivial in this setting. With a strictly monotone index, we can calculate the number of successes and failures in the full history that leads to stopping given a fixed history length. Though we can not pin down the path of the history, it suffices to compute the correct posterior. Similar to Sethi and Yildiz (2016), which models the learning on both agents' perspectives and information, in the case where the stopping rule is not fully public, there is a nested learning on both the underlying random payoff and the index. Stopping reveals information about the index. For instance, an insider who stops right away after one failure following one success must have a stopping rule that is more sensitive to failures than the one who stops after one success and two failures. This is a direct sequel that I attempt to work on.

For tractability, I did not consider strategic interaction between agents, but the insider's stopping rule is often distorted by signaling purposes. For instance, Halac and Kremer (2020) and Thomas (2019) show that agents tend to over-experiment if admitting failure harms their reputation. The insider's strategic feedback of using a less harsh stopping rule attenuates the bias. It is also possible to consider information disclosure as in Ben-Porath et al. (2018), where agents voluntarily reveal private signals, resulting in an adverse selection problem. This model can also extend to a multi-agent learning setting as in the information herding framework. The posterior from observing a predecessor forms the prior of the successor, reinforcing the bias.

6 Appendix

I first introduce some terminology and computations that will be used in the proofs. If prehistory H^- is admissible, it enters the computation of the sophisticated posterior as $\sigma(H^-)$ and $\varphi(H^-)$. So I first characterize the space of (σ, φ). For an arbitrary history $\langle s_t \rangle_{t=-T}^{z}$, I say $(\sigma(\langle s_t \rangle_{t=-T}^{z}), \varphi(\langle s_t \rangle_{t=-T}^{z})) \in \mathbb{N}^2$ is a state in the insider's stopping problem. The full history $\langle s_t \rangle_{t=-T}^{Z}$ forms a path through the states $(\sigma(\langle s_t \rangle_{t=-T}^{z}), \varphi(\langle s_t \rangle_{t=-T}^{z}))$, $z = -T, \ldots, Z$. Accounting for all possible prehistories is equivalent to considering all the possible states whose weights are determined by the number of admissible pathes leading to them. Let $\Omega_H \subseteq \mathbb{N}^2$ denote the set of historical states corresponding to the pathes of all H-admissible histories,

$$\Omega_H(I, T) = \cup_{z=-T}^{-1} \{ (\sigma(\langle s_t \rangle_{t=-T}^{z}), \varphi(\langle s_t \rangle_{t=-T}^{z})) | \langle s_t \rangle_{t=-T}^{-1} \in \Gamma_H(I, T) \} .$$

Assume a state (σ, φ) is the σ, φth vertex in a 2-dimensional grid. Since $I(\cdot)$ weakly rises in $\sigma(\cdot)$ and falls in $\varphi(\cdot)$, we have that $(\sigma, \varphi) \in \Omega_H(I, T)$ implies $(\sigma + a, \varphi - b) \in \Omega_H(I, T)$ where $a, b \geq 0$ and $a + b \leq T - \sigma - \varphi$. On the other hand, $(\sigma, \varphi) \notin \Omega_H(I, T)$ implies $(\sigma - a, \varphi + b) \notin \Omega_H(I, T)$ for any $a, b \geq 0$. Hence, $\Omega_H(I, T)$ is a quasi-upper triangular region in the space of $\mathbb{N}^2$.

I say a sequence of states $\{(\sigma_1, \varphi_1), \ldots, (\sigma_K, \varphi_K)\}$ is a path of staircase walk if $\sigma_{k+1} \in \{\sigma_k, \sigma_k + 1\}$ and $\varphi_{k+1} \in \{\varphi_k, \varphi_k + 1\}$ for $k = 1, \ldots, K - 1$. Let $N(\sigma, \varphi | \Omega)$ be the number of staircase walks from $(0, 0)$ to (σ, φ) whose paths are entirely inside Ω. The number of H-admissible prehistories with σ successes is exactly $N(\sigma, T - \sigma | \Omega_H(I, T))$. Thus the proportional density (3) can be written as

$$f_p(H; I, T) = p^{\sigma(H)}(1 - p)^{\varphi(H)} \sum_{\sigma=0}^{T} N(\sigma, T - \sigma | \Omega_H(I, T)) p^{\sigma}(1 - p)^{T - \sigma} .$$

Next, I show that the staircase walk number $N(\sigma, \varphi | \Omega)$ can be recursively computed. If a full history has only successes or only failures, the path is unique. Thus $N(\sigma, 0 | \Omega) = N(0, j | \Omega) = 1$ when $(\sigma, 0)$, $(0, \varphi) \in \Omega$. When $(\sigma, \varphi) \notin \Omega$, we have $N(\sigma, \varphi | \Omega) = 0$. Since a path can only reach state (σ, φ) from $(\sigma - 1, \varphi)$ or $(\sigma, \varphi - 1)$ when $\sigma, \varphi \geq 1$, the remaining cases where $\sigma, \varphi \geq 1$ and $(\sigma, \varphi) \in \Omega_H(I, T)$ can be derived by the following rules

$$N(\sigma, \varphi | \Omega) = N(\sigma, \varphi - 1 | \Omega) + N(\sigma - 1, \varphi | \Omega) . \tag{8}$$

Note that the sophisticated posterior is the weighted average of a sequence of distributions whose densities are proportional to $p^{\sigma}(1 - p)^{T - \sigma}$. These distributions are ranked in the likelihood ratio order $\succeq_{\text{lr}}$ according to σ. This fact is critical more many of my results. I first show that this kind of compositional distributions rises in $\succeq_{\text{lr}}$ if the relative weighs on its composition that's higher in $\succeq_{\text{lr}}$ becomes larger.

Claim 1. *Suppose a sequence of distributions $F^1 ... F^n$ have differentiable densities*[14] $f_1, \ldots, f_n$ *and satisfy $F^j \succeq_{lr} F^i$ for $\forall i < j$. Let $G^A = \sum_{i=0}^n a_i F^i$ and $G^B = \sum_{i=0}^n b_i F^i$, where $\sum_{i=1}^n a_i = \sum_{i=1}^n b_i = 1$, $a_i, b_i \geq 0$. If*[15] $a_j/a_i \leq b_j/b_i$ *for all $j \geq i$, we have $G^B \succeq_{lr} G^A$.*

Proof. We need to show that $\sum_{i=1}^n a_i f_i' / \sum_{i=1}^n a_i f_i \leq \sum_{i=1}^n b_i f_i' / \sum_{i=1}^n b_i f_i$, or equivalently, $\sum_{i=1}^n \sum_{j=1}^n a_i f_i' b_j f_j \leq \sum_{i=1}^n \sum_{j=1}^n a_i f_i b_j f_j'$. Cancel out terms in which $i = j$, this becomes

$$\sum_{i>j} a_i f_i' b_j f_j + \sum_{i<j} a_i f_i' b_j f_j \leq \sum_{i>j} a_i f_i b_j f_j' + \sum_{i<j} a_i f_i b_j f_j' ,$$

which simplifies to

$$\sum_{i>j} a_i b_j \left(f_i' f_j - f_i f_j' \right) \leq \sum_{i<j} a_i b_j \left(f_i f_j' - f_i' f_j \right) .$$

Exchanging the dummies in the LHS, the goal becomes showing the following,

$$\sum_{i<j} a_j b_i \left(f_j' f_i - f_j f_i' \right) \leq \sum_{i<j} a_i b_j \left(f_i f_j' - f_i' f_j \right) . \tag{9}$$

Since $F_j \succeq_{\text{lr}} F_i$ for $\forall i \leq j$, we have $f_j'/f_j > f_i'/f_i$ for $\forall i \leq j$. By assumption, we have $a_j b_i > a_i b_j$ for $\forall i < j$. Therefore, $a_j b_i \left(f_i f_j' - f_i' f_j \right) < a_i b_j \left(f_i f_j' - f_i' f_j \right)$ for $\forall i < j$ and thus (9) holds. □

Proof of Lemma 1

To show that $F(H; I, T)$ stochastically rises when the index I is harsher , I first show that the relative weights on states with more successes gets higher when we exclude more prehistories. The rest follows from Claim 1. If I' is harsher than I, then the H-admissible prehistories $\Gamma_H(I', T) \subseteq \Gamma_H(I, T)$, and thus $\Omega_H(I', T) \subseteq \Omega_H(I, T)$. Next, I show that when $\sigma' > \sigma$, the relative weight $N(\sigma', T - \sigma'|\Omega)/N(\sigma, T - \sigma|\Omega)$ falls in the size of Ω in the following sense:

Claim 2. *If $\tilde{\Omega} = \Omega \cup \{(\hat{\sigma}, \hat{\varphi})\}$ where $\hat{\sigma} + \hat{\varphi} \leq T$, we have that for $\forall\ \sigma < \sigma' \leq T$ and $N(\sigma, T - \sigma|\Omega)$, $N(\sigma, T - \sigma|\tilde{\Omega}) > 0$,*

$$N(\sigma', T - \sigma'|\tilde{\Omega})/N(\sigma, T - \sigma|\tilde{\Omega}) \leq N(\sigma', T - \sigma'|\Omega)/N(\sigma, T - \sigma|\Omega) .$$

Proof. I show the equivalence

$$N(\sigma', T - \sigma'|\tilde{\Omega}) N(\sigma, T - \sigma|\Omega) \leq N(\sigma', T - \sigma'|\Omega) N(\sigma, T - \sigma|\tilde{\Omega}) . \tag{10}$$

Assume that $(\hat{\sigma}, \hat{\varphi})$ is on the lower boundary of Ω, namely, $(\hat{\sigma}, \hat{\varphi}) \notin \Omega$, $(\hat{\sigma}+1, \hat{\varphi}-1) \in \Omega$, and $(\hat{\sigma}, \hat{\varphi}-1) \in \Omega$. I show this claim by induction on the position of states (σ, φ).

[14] Here, subscripts are numerators though they usually denote the argument of f and F elsewhere in this paper. I use the prime symbol as the derivative notation in the proof of this claim.

[15] The case where a_i or b_i are 0 trivially satisfy the desired property and I omit the case.

Step 1. I show

$$N(\sigma', t-\sigma'|\tilde{\Omega})N(\sigma, t-\sigma|\Omega) \le N(\sigma', t-\sigma'|\Omega)N(\sigma, t-\sigma|\tilde{\Omega}) \ . \tag{11}$$

for any $\sigma < \sigma'$ and $t = \hat{\sigma} + \hat{\varphi} \le T$. Since $N(\sigma, t-\sigma|\Omega) = N(\sigma, t-\sigma|\tilde{\Omega}) = 0$ for all $\sigma < \hat{\sigma}$, $0 = N(\hat{\sigma}, t-\sigma|\Omega) < N(\hat{\sigma}, t-\hat{\sigma}|\tilde{\Omega})$, and $N(\sigma, t-\sigma|\Omega) = N(\sigma, t-\sigma|\tilde{\Omega}) > 0$ for all $\sigma > \hat{\sigma}$. Given these, the following four cases completes the proof of (11) for $t = \hat{\sigma} + \hat{\varphi}$:

(1) $\sigma' > \sigma > \hat{\sigma}$. We have $N(\sigma', t-\sigma'|\tilde{\Omega})N(\sigma, t-\sigma|\Omega) = N(\sigma', t-\sigma'|\Omega)N(\sigma, t-\sigma|\tilde{\Omega}) > 0$;

(2) $\sigma' > \sigma = \hat{\sigma}$. We have $0 = N(\sigma', t-\sigma'|\tilde{\Omega})N(\sigma, t-\sigma|\Omega) < N(\sigma', t-\sigma'|\Omega)N(\sigma, t-\sigma|\tilde{\Omega})$;

(3) $\hat{\sigma} \ge \sigma' > \sigma$. We have $N(\sigma', t-\sigma'|\tilde{\Omega})N(\sigma, t-\sigma|\Omega) = N(\sigma', t-\sigma'|\Omega)N(\sigma, t-\sigma|\tilde{\Omega}) = 0$;

(4) $\sigma' > \hat{\sigma} \ge \sigma$. We have $0 = N(\sigma', t-\sigma'|\tilde{\Omega})N(\sigma, t-\sigma|\Omega) \le N(\sigma', t-\sigma'|\Omega)N(\sigma, t-\sigma|\tilde{\Omega})$.

Step 2. For arbitrary σ and $t = \hat{\sigma} + \hat{\varphi}$, let $A_k = N(\sigma+k, t-\sigma-k|\Omega)$ and $B_k = N(\sigma+k, t-\sigma-k|\tilde{\Omega})$ for $k = 1, 2, 3$. By step 1, we have $A_1B_2 \le B_1A_2$, $A_1B_3 < B_1A_3$ and $A_2B_3 < B_2A_3$, thus $(A_1 + A_2)(B_2 + B_3) \le (B_1 + B_2)(A_2 + A_3)$. By (8), we have $A_k + A_{k+1} = N(\sigma+k+1, t-\sigma-k|\Omega)$ and $B_k + B_{k+1} = N(\sigma+k+1, t-\sigma-k|\tilde{\Omega})$. Hence (11) holds for $t = \hat{\sigma}+\hat{\varphi}+1$. By induction on t, (11) holds for $t = T$, thus we have (10). $\square$

The admissible states set is quasi-upper-triangular shaped. Expanding it only includes more states of high failure and low success numbers. Moreover, the relative weights on the bad states get higher. The rest follows from Claim 1.

Proof of Proposition 1

Note that $\Gamma_H(I, T) \subseteq \Gamma_\varnothing(I, T)$ for any H, it suffices to show $F(\varnothing; I, T) \succeq_{\text{lr}} G(\tilde{H})$.

Step 1. First consider $I(\cdot) = \sigma(\cdot) - a\phi(\cdot)$ where $a \ge 1$ is an interger.Similar to (4), we have $N(T-m, m|\Omega_\varnothing(I, T)) = c(m, \rho, T+1-\rho m)$ where. Thus

$$f_p(\varnothing; I, T) = \textstyle\sum_{m=0}^{\lceil T/(a+1)\rceil} c(m, a+1, T+1-(a+1)m)p^{T-m}(1-p)^m\pi(p) \ . \tag{12}$$

Denote the probability density of $G(\cdot)$ as $g_p(\cdot)$. Note that for any $\tilde{H}$ and t,

$$g_p(\tilde{H}) = p^{\sigma(\tilde{H})}(1-p)^{\varphi(\tilde{H})}\textstyle\sum_{n=0}^{t}\binom{t}{n}p^{t-n}(1-p)^n\pi(p) \ . \tag{13}$$

Let $T' = T - \sigma(\tilde{H}) - \varphi(\tilde{H})$. Plug in $\varphi(\tilde{H}) = 1$ and $\sigma(\tilde{H}) = 4a$, I rewrite it as

$$g_p(\tilde{H}) = \textstyle\sum_{n=0}^{T'}\binom{T'}{n}p^{T'-n+4a}(1-p)^{n+1}\pi(p) = \sum_{n=0}^{T'}\binom{T'}{n}p^{T-n-1}(1-p)^{n+1}\pi(p) \ .$$

By Claim 1, it suffices to show that for each $1 \le m = n + \varphi(\tilde{H}) \le \lceil T/(a+1)\rceil$,

$$\frac{c(m-1, a+1, T+1-(a+1)(m-1))}{c(m, a+1, T+1-(a+1)m)} \ge \binom{T'}{n-1} \Big/ \binom{T'}{n} \ .$$

Plugging the expression of $c(m,\rho,r)$, it simplifies to

$$(T+1)(n-T)/\left[(n+2)(n(a+1)+a-T)\right] > 4a \; .$$

The LHS of the above inequality is convex in n if $n \leq \lceil T/(a+1) \rceil$, and reaches minimum at $n = T - \sqrt{(T+1)(T+2)a/(a+1)}$. Plug in this and denote its minimum as $r(T)$,

$$r(T) = (T+1)\left(3a+2-1+(2a+1)T+2\sqrt{a(a+1)(T+1)(T+2)}\right)/(a+T+2)^2 \; .$$

By some algebra, we have $r(T) \geq 4a$ if $T \geq (2a+1)^2 - 2$, thus complete step 1.

Note that $\lim_{T\to\infty} r(T) = 2a+1+2\sqrt{a(a+1)}$, which is greater than $4a+1$ when $a \geq 0$. It is easy to show that $r(T)$ rises in T. We have $r(T) > 4a+1$ as long as $T \geq (a+1)\sqrt{(4a+1)^3/a}/2 + 4(a+2)a$, where we derive the tighter bound

$$F(\varnothing; I, T) \succeq_{\mathrm{lr}} G((\langle 1\{4a+1\}0\rangle) \; .$$

Before moving on, I show the next result that will be used in the proof of both this proposition and Proposition 4. For simplicity, let $N_{i,j}$ denote $N(i,j|\Omega_H(I,T))$.

Claim 3. *For all* $i+j \leq T$*,*

$$N_{i+1,j-1}/N_{i,j} \leq N_{i+1,j}/N_{i,j+1} \leq N_{i,j}/N_{i-1,j+1} \; . \tag{14}$$

Proof. Note that $N_{i,j} = 0$ implies $(i,j) \notin \Omega_H(I,T)$, and thus $N_{i-t,j} = N_{i,j+t} = 0$ for any $t \geq 0$. Therefore if any of the terms in (14) is 0, it holds trivially. Thus I consider only the case where all the terms are positive. By (8), we have $N_{i+1,j} = N_{i,j} + N_{i+1,j-1}$, and $N_{i+1,j} = N_{i,j} + N_{i+1,j-1}$. Plugg these into (14) and modify the dummies, both inequalities simplify to

$$N_{i,j}^2 \geq N_{i-1,j+1} N_{i+1,j-1} \; . \tag{15}$$

I show this by induction on $K := i+j$.
(1) Extend the domain of N_{ij} from $\mathbb{N}^2$ to $\mathbb{Z}^2$ and set $N_{i,j} = 0$ for all $(i,j) \notin \Omega$. Therefore, we have $N_{i,-i}^2 \geq N_{i+1,-(i+1)} N_{(i-1),-(i-1)}$ for all i since $N_{0,0} = 1$ and $N_{i,-i} = 0$ for all $i \neq 0$. Thus we transform (15) into the following: for all i and $K = 0$,

$$N_{i,K-i}^2 \geq N_{i+1,K-(i+1)} N_{(i-1),K-(i-1)} \; . \tag{16}$$

(2) If (16) holds for all i and some $K = k \geq 0$, I show that it holds for all i and $K = k+1$. Still, I focus on the case where $N_{i-1,k+2-i}$, $N_{i,k+1-i}$, and $N_{i+1,k-i} > 0$. By (8), we have $N_{i+1,k-i} = N_{i,k-i} + N_{i+1,k-i-1}$, $N_{i,k+1-i} = N_{i-1,k+1-i} + N_{i,k-i}$ and $N_{i-1,k+2-i} = N_{i-2,k+2-i} + N_{i-1,k+1-i}$. Plugging in these relations and applying the assumption that $N_{i-1,k+1-i}^2 \geq N_{i-2,k+2-i} N_{i,k-i}$ and $N_{i,k-i}^2 \geq N_{i-1,k+1-i} N_{i+1,k-1-i}$, we have $N_{i,k+1-i}^2 \geq N_{i+1,k-i} N_{i-1,k+2-i}$. Therefore, (16) holds for all i and $K = k+1$.

By induction on K, (15) holds for all $i+j \leq T$. Thus (14) holds for all $i+j \leq T$. □

Step 2. Next, consider $I^C(\cdot) = \sigma(\cdot) - a\phi(\cdot) + C$ where $a \geq 1$ and $C \geq 0$ are integers. Let $N^C_{i,j}$ denote $N(i,j|\Omega_\varnothing(I^C, T))$, we have

$$f_p(\varnothing; I^C, T) = \sum_{m=0}^{\lceil T/(a+1)\rceil} N^C_{T-m,m} p^{T-m}(1-p)^m \pi(p) \ .$$

Similar to Step 1, I show that for any C and $1 \leq m = n + C + 1 \leq \lceil T/(a+1)\rceil$,

$$N^C_{T-m+1,m-1}/N^C_{T-m,m} \geq \binom{T'}{n-1}/\binom{T'}{n} \ . \tag{17}$$

where $T' = T-(4a-C)-1$ for case (a) and $T' = T-4a-\lceil C/a\rceil-1$ for (b). The instance that $C = 0$ is shown in Step 1. I proceed by induction. Assume (17) holds for $C = c$. For case (a), note that $N^{c+1}_{i-1,j} = N(i,j|\Omega')$ where $\Omega' = \Omega_\varnothing(I^c, T)\setminus \cup_{\varphi=0}^{\lfloor c/(a+1)\rfloor} \{(0,\varphi)\}$. Let $N'_{i,j} := N(i,j|\Omega')$. By step 1 of Claim 2, $N'_{i+1,j}/N'_{i,j+1} \geq N^c_{i,j}/N^c_{i-1,j+1}$. By Claim 3, we have $N^c_{i,j}/N^c_{i-1,j+1} \geq N^c_{i+1,j}/N^c_{i,j+1}$. (a) follows from

$$\frac{N^{c+1}_{T-m+1,m-1}}{N^{c+1}_{T-m,m}} = \frac{N'_{T-m,m-1}}{N'_{T-m-1,m}} \geq \frac{N^c_{T-m,m-1}}{N^c_{T-m-1,m}} \geq \frac{N^c_{T-m+1,m-1}}{N^c_{T-m,m}} \geq \binom{T'}{n-1}/\binom{T'}{n};$$

(b) $N^{c+1}_{i+1,j-1} = N(i,j|\Omega'') := N''_{i,j}$ where $\Omega'' = \Omega_\varnothing(I^c, T)\setminus \cup_{\sigma=0}^{a-1}\left(\cup_{\varphi=0}^{\lfloor c/(a+1)\rfloor}\{(\sigma,\varphi)\}\right)$. Similarly, we have $N''_{i+1,j}/N''_{i,j+1} \geq N^c_{i,j}/N^c_{i-1,j+1}$ and $N^c_{i,j}/N^c_{i-1,j+1} \geq N^c_{i,j-1}/N^c_{i-1,j}$. Thus

$$\frac{N^{c+1}_{T-m+1,m-1}}{N^{c+1}_{T-m,m}} = \frac{N''_{T-m,m}}{N''_{T-m-1,m+1}} \geq \frac{N^c_{T-m,m}}{N^c_{T-m-1,m+1}} \geq \frac{N^c_{T-m+1,m-1}}{N^c_{T-m,m}} \geq \binom{T'}{n-1}/\binom{T'}{n}.$$

The results when a and c are nonintegers follow easily from Lemma 1.

Asymptotic Property of the Sophisticated Posterior

Proposition 5. *If our prior has full support and $I(\cdot)$ is harsher than $\sigma(\cdot) - a\varphi(\cdot)$ where $a \geq 1$, the sophisticated posterior puts probability 1 on $P > \lfloor a\rfloor/(\lfloor a\rfloor + 1)$ as T approaches ∞.*

Proof. First assume a is a positive integer. Plug $c(m,\rho,r) = \frac{r}{m\rho+r}\binom{m\rho+r}{m}$ into (12),

$$\begin{aligned} f_p(\varnothing; I, T) &= \sum_{m=0}^{\lceil T/(a+1)\rceil} \frac{T+1-(a+1)m}{T+1}\binom{T+1}{m} p^{T-m}(1-p)^m\pi(p) \\ &= \sum_{m=0}^{\lceil T/(a+1)\rceil} \frac{T+1-(a+1)m}{T+1-m}\binom{T}{m} p^{T-m}(1-p)^m\pi(p) \ . \end{aligned}$$

By some algebra, $\lim_{T\to\infty} f_p(\varnothing; I, T) = ((a+1)p - a)\,\pi(p)$ when $p > a/(a+1)$. On the other hand, $\lim_{T\to\infty} f_p(\varnothing; I, T) = 0$ when $p \leq a/(a+1)$. Since the prior has full support, $\pi(p) > 0$ for $\forall p \in [0,1]$. Integrating f, we have the desired result for $H = \varnothing$.

Since $\Gamma_H(I,T) \subseteq \Gamma_\varnothing(I,T)$ for general H, by Lemma 1, $F(H;I,T)$ is higher in the likelihood ratio order than some $\hat{F}$ with probability density

$$\hat{f}_p = \sum_{m=0}^{\lceil T/(a+1)\rceil} \frac{T+1-(a+1)m}{T+1-m}\binom{T}{m} p^{T-m+\sigma(H)}(1-p)^{m+\varphi(H)}\pi(p).$$

Similarly, we have that $\lim_{T\to\infty} \hat{f}_p = ((a+1)p-a)\, p^{\sigma(H)}(1-p)^{\varphi(H)}\pi(p)$ for $p > a/(a+1)$ and $\lim_{T\to\infty} \hat{f}_p = 0$ for $p \le a/(a+1)$. Since being higher in the likelihood ratio order implies first order stochastic dominance, the result holds for general public histories. The case where a is a noninteger also follows easily from Lemma 1. □

Proof of Proposition 2

First, I first show part (b), and then part (a) and (c).
(b) If H is indecisive, the admissible prehistories $\Gamma_H(I,T) = \Gamma_\varnothing(I,T)$. Hence the historical states $\Omega_H(I,t) = \Omega_\varnothing(I,T)$ and $f_p(H;I,t) = p^{\sigma(H)}(1-p)^{\varphi(H)} f_p(\varnothing;I,T)$. If H is noncritical, then H must also be indecisive at $T+1$. By (3), the proportional densities satisfy

$$f_p(\varnothing;I,T) = \sum_{i=0}^{t} N(i,T-i|\Omega_\varnothing(I,T))p^i(1-p)^{T-i}\,; \tag{18}$$
$$f_p(\varnothing;I,T+1) = \sum_{i=0}^{t} N(i,T+1-i|\Omega_H(I,T+1))p^i(1-p)^{T+1-i}\,. \tag{19}$$

By the relation of the admissible prehistory sets (7), the admissible states satisfy

$$\Omega_\varnothing(I,T+1) = \cup_{i=\hat\sigma}^{T}\{(i,T+1-i)\} \cup \Omega_\varnothing(I,T)\,, \tag{20}$$

where $\hat\sigma = \sigma(\tilde{H}^-(I,T))$ is the success number of the worst prehistory in $\Gamma_\varnothing(I,T)$. In the following arguments, I use $N_{i,j}$ in short for $N(i,j|\Omega_\varnothing(I,T+1))$. Note that if $(i,j) \in N(i,j|\Omega_\varnothing(I,T))$, we also have $N(i,j|\Omega_\varnothing(I,T)) = N_{i,j}$. The RHS of (18) satisfies

$$\begin{aligned}\sum_{i=0}^{T} N_{i,T-i}\, p^i(1-p)^{T-i} &= \sum_{i=\hat\sigma}^{T} N_{i,T-i}\, p^i(1-p)^{T-i} = \sum_{i=\hat\sigma}^{T} N_{i,T-i}\, p^i(1-p)^{T-i}(p+1-p)\\ &= \sum_{i=\hat\sigma}^{T} N_{i,T-i}\, p^{i+1}(1-p)^{T-i} + \sum_{i=\hat\sigma}^{T} N_{i,T-i}\, p^i(1-p)^{T+1-i}\,. \end{aligned}\tag{21}$$

Next, plugging in the relations by (8), $N_{i,T+1-i} = N_{i-1,T+1-i} + N_{i,T-i}$ for any $\hat\sigma < i < T+1$, $N_{\hat\sigma,T+1-\hat\sigma} = N_{\hat\sigma,T-\hat\sigma}$ and $N_{T+1,0} = N_{T,0}$, the RHS of (19) becomes:

$$\begin{aligned}&\sum_{i=0}^{T+1} N_{i,T+1-i}\, p^i(1-p)^{T+1-i} = \sum_{i=\hat\sigma}^{T+1} N_{i,T+1-i}\, p^i(1-p)^{T+1-i}\\ =&\sum_{i=\hat\sigma+1}^{T} N_{i,T+1-i}\, p^i(1-p)^{T+1-i} + N_{\hat\sigma,T+1-\hat\sigma}\, p^{\hat\sigma}(1-p)^{T+1-\hat\sigma} + N_{T+1,0}\, p^T\\ =&\sum_{i=\hat\sigma+1}^{T} (N_{i,T-i} + N_{i-1,T+1-i})\, p^i(1-p)^{T+1-i} + N_{\hat\sigma,T-\hat\sigma}\, p^{\hat\sigma}(1-p)^{T+1-\hat\sigma} + N_{T,0}\, p^{T+1}\,.\end{aligned}$$

Distribute the terms in the first product and merge them with the last two terms separately, we find the results equal to the two sums in (21) respectively:

$$\textstyle\sum_{i=\hat\sigma+1}^{T} N_{i,T-i}\, p^i(1-p)^{T+1-i} + N_{\hat\sigma,T-\hat\sigma}\, p^{\hat\sigma}(1-p)^{T+1-\hat\sigma} = \sum_{i=\hat\sigma}^{T} N_{i,T-i}\, p^i(1-p)^{T+1-i}\,; \quad (22)$$

$$\begin{aligned}\textstyle\sum_{i=\hat\sigma+1}^{T} N_{i-1,T+1-i}\, p^i(1-p)^{T+1-i} + N_{T,0}\, p^{T+1} &= \textstyle\sum_{i=\hat\sigma}^{T-1} N_{i,T-i}\, p^{i+1}(1-p)^{T-i} + N_{T,0}\, p^{T+1} \\ &= \textstyle\sum_{i=\hat\sigma}^{T} N_{i,T-i}\, p^{i+1}(1-p)^{T-i}\,. \quad (23)\end{aligned}$$

By (21) to (23), the RHS of (18) and (19) are equivalent, thus $f_p(\varnothing; I, T+1) = f_p(\varnothing; I, T)$.

(a) If H is critical, by (7), the relation (20) changes into

$$\Omega_H(I, T+1) = \cup_{i=\hat\sigma+1}^{T}\{(i, T+1-i)\} \cup \Omega_\varnothing(I, T) \subset \cup_{i=\hat\sigma}^{T}\{(i, T+1-i)\} \cup \Omega_\varnothing(I, T)\,.$$

Part (a) shows that $F(H; I, T+1) \equiv F(H; I, T)$ if $\Omega_H(I, T+1) = \cup_{i=\hat\sigma}^{T}\{(i, T+1-i)\} \cup \Omega_H(I, T)$. By Claim 2, now we have $F_p(H; I, T+1) \succeq_{\text{lr}} F_p(H; I, T)$.

Note that the equation (20) is critical for our analysis. When H is decisive and non-critical, the simple equivalence and subset relations fail and the previous arguments do not go through.

(c) Now H is decisive and noncritical, the H-admissible prehistory sets satisfy

$$\{H^- + \langle 1\rangle, H^- + \langle 0\rangle | H^- \in \Gamma_H(I, T)\} \subset \Gamma_H(I, T+1)\,.$$

In particular, there are more paths leading to the worst state $(\hat\sigma, T-\hat\sigma)$ in $\Omega_\varnothing(I, T+1)$ than in $\Omega_\varnothing(I, T)$ where $\hat\sigma = \min_{H^- \in \Gamma_H(I, T+1)} \sigma(H^-)$. By Claim 1, we have $F(H; I, T) \succeq_{\text{lr}} F(H; I, T+1)$.

Proof of Proposition 3

I prove this proposition by first showing that the H'-admissible prehistory set is a subset of the H-admissible prehistory set if H' has earlier failure. The rest follows from Lemma 2.

Let $H = \langle s_t\rangle_{t=0}^{Z}$ and $H' = \langle s'_t\rangle_{t=0}^{Z}$. For any H^- in the H'-admissible prehistory set, we have $I(H^- + \langle s'_t\rangle_{t=0}^{z}) \geq 0$, for all $z = -T, \ldots, -1$. Since H' has earlier failures than H, we have $\sigma(\langle s'_t\rangle_{t=0}^{z}) \leq \sigma(\langle s_t\rangle_{t=0}^{z})$ for all $z = 0, \ldots, Z$. By the monotonicity of I, we have

$$I(H^- + \langle s_t\rangle_{t=0}^{z}) \geq I(H^- + \langle s'_t\rangle_{t=0}^{z}) \geq 0$$

for all $z = 0, \ldots, Z$. Thus H^- is also in the H-admissible prehistory set, namely $\Gamma_{H'}(I, T) \subseteq \Gamma_H(I, T)$ and $\Omega_{H'}(I, T) \subseteq \Omega_H(I, T)$.

Next, we apply Claim 2. The weights on states with more successes and fewer failures are higher when the public history has earlier failures. Finally apply Claim 1, we have $F_p(H'; I, T) \succeq_{\text{lr}} F_p(H; I, T)$ when H' has earlier failures than H.

Proof of Proposition 4

I first show the special case of (a) where $t = 1$, namely

$$F(\langle 1\rangle + H; I, T-1) \succeq_{\text{lr}} F(H; I, T) \succeq_{\text{lr}} F(\langle 0\rangle + H; I, T-1).$$

Next, I prove part (b), then the general result of (a) follows natually.
(a) By (3), we have

$$\begin{aligned} f_p(H; I, T) &= p^{\sigma(H)}(1-p)^{\varphi(H)} \textstyle\sum_{i=0}^{T} N(i, T-i|\Omega_H(I,T)) p^i (1-p)^{T-i} \\ f_p(\langle 1\rangle + H; I, T-1) &= p^{\sigma(H)+1}(1-p)^{\varphi(H)} \textstyle\sum_{i=1}^{T-1} N(i-1, T-i|\Omega_H(I,T)) p^{i-1} (1-p)^{T-i} \\ f_p(\langle 0\rangle + H; I, T-1) &= p^{\sigma(H)}(1-p)^{\varphi(H)+1} \textstyle\sum_{i=0}^{T-1} N(i, T-i-1|\Omega_H(I,T)) p^i (1-p)^{T-i-1}. \end{aligned}$$

By Claim 3, the relative weights on states with more successes and fewer failures are higher, the rest follows from Claim 1.

(b) Case 1. First consider the case that $s_{-t} = 1$. Note that that for any $(i,j) \in \Omega_H(I,T)$, the number of staircase walks from $(0,0)$ to (i,j) within $\Omega_H(I,T)$ that contains the step from (σ, φ) to $(\sigma+1, \varphi)$ can be computed by multiplying $N_{\sigma,\varphi}$ and the number of staircase walks from $(\sigma+1, \varphi)$ to (i,j) within $\Omega_H(I,T)$, which I denote as $\tilde{N}_{i,j}^{\sigma+1,\varphi}$. Let $N'_{i,j}$ be the number of staicase walks from $(0,0)$ to (i,j) within $\Omega_H(I,T)$ that contains the step from (σ, φ) to $(\sigma+1, \varphi)$ for any $\sigma + \varphi = T - t$. For any $i + j = T$ and $i \geq \sigma(\tilde{H}^-(I,T))$, we can compose the expression of $N'_{i,j}$ as the following,

$$N'_{i,j} = \begin{cases} \sum_{k=0}^{j} N_{i-t+k, j-k} \tilde{N}_{i,j}^{i'-t+k, j-k} & i \geq \hat{\sigma} \\ \sum_{k=\hat{\sigma}-t}^{i-1} N_{k, T-t-k} \tilde{N}_{i,j}^{k+1, T-t-k} & i \leq \hat{\sigma} \end{cases}, \tag{24}$$

where $(\hat{\sigma} - t, T - \hat{\sigma}) \in \Omega_H(I,T)$ and $(\hat{\sigma} - t - 1, T - \hat{\sigma} + 1) \notin \Omega_H(I,T)$.

Let $\tilde{f}_p(H; I, T|s_{-t} = 1)$ denote the probability density function of the sophisticated posterior with the information that $s_{-t} = 1$, so that

$$\tilde{f}_p(H; I, T|s_{-t} = 1) = p^{\sigma(H)}(1-p)^{\varphi(H)} \sum_{i=\hat{\sigma}}^{T} N'_{i, T-i} p^i (1-p)^{T-i}.$$

Let $t' = t + 1$, we denote the number of staircase walks from $(0,0)$ to (i,j) within $\in \Omega_H(I,T)$ that contain the path from (σ, φ) to $(\sigma+1, \varphi)$ for any $\sigma + \varphi = T - t'$ as $N''_{i,j}$. Similarly to (24), we have

$$N''_{i,j} = \begin{cases} \sum_{k=0}^{j} N_{i-t'+k, j-k} \tilde{N}_{i,j}^{i'-t'+k, j-k} & i \geq \hat{\sigma}' \\ \sum_{k=\hat{\sigma}'-t'}^{i-1} N_{k, T-t'-k} \tilde{N}_{i,j}^{k+1, T-t'-k} & i \leq \hat{\sigma}' \end{cases}, \tag{25}$$

where $(\hat{\sigma}' - t', T - \hat{\sigma}') \in \Omega_H(I,T)$ and $(\hat{\sigma}' - t' - 1, T - \hat{\sigma}' + 1) \notin \Omega_H(I,T)$. Note that we have either $\hat{\sigma}' = \hat{\sigma}$ or $\hat{\sigma}' = \hat{\sigma} - 1$ and

$$\tilde{f}_p(H; I, T | s_{-t'} = 1) = p^{\sigma(H)}(1-p)^{\varphi(H)} \sum_{i=\hat{\sigma}}^{T} N''_{i,T-i} p^i (1-p)^{T-i} .$$

It suffices to show that $\tilde{f}_p(H; I, T | s_{-t'} = 1) = \tilde{f}_p(H; I, T | s_{-t} = 1)$.

Case 1-i. $\hat{\sigma}' = \hat{\sigma}$. In this case we must have $(\hat{\sigma}' - t', T - \hat{\sigma}' + 1) \notin \Omega_H(I, T)$. Note that for each $\sigma \le i - 1$ and $\varphi \le j - 1$, we have $\tilde{N}^{\sigma,\varphi}_{i,j} = \tilde{N}^{\sigma+1,\varphi}_{i,j} + \tilde{N}^{\sigma,\varphi+1}_{i,j}$, also note that $N_{0,\varphi} = N_{\sigma,0} = 1$ for any σ, φ and $k \ge 0$. Let $i' = i + 1$, $j' = j - 1$ and $T' = T + 1$. Applying these facts and (8), we rearrange (25) for $i \ge \hat{\sigma}'$:

$$\begin{aligned}
N''_{i,j} &= \textstyle\sum_{k=1}^{j} N_{i\text{-}t'+k,j\text{-}k}(\tilde{N}^{i'\text{-}t+k,j\text{-}k}_{i,j} + \tilde{N}^{i\text{-}t+k,j\text{-}k+1}_{i,j}) + N_{i\text{-}t',j} \\
&= \textstyle\sum_{k=1}^{j} N_{i\text{-}t'+k,j\text{-}k}\tilde{N}^{i'\text{-}t+k,j\text{-}k}_{i,j} + \sum_{k=0}^{j'} N_{i\text{-}t+k,j'\text{-}k}\tilde{N}^{i'\text{-}t+k,j\text{-}k}_{i,j} + N_{i\text{-}t',j} \\
&= \textstyle\sum_{k=1}^{j'} N_{i\text{-}t'+k,j\text{-}k}\tilde{N}^{i'\text{-}t+k,j\text{-}k}_{i,j} + \tilde{N}^{i'\text{-}t+j,0}_{i,j} + \sum_{k=1}^{j'} N_{i\text{-}t+k,j'\text{-}k}\tilde{N}^{i'\text{-}t+k,j\text{-}k}_{i,j} + N_{i\text{-}t,j'} + N_{i\text{-}t',j} \\
&= \textstyle\sum_{k=1}^{j'} \left(N_{i\text{-}t'+k,j\text{-}k} + N_{i\text{-}t+k,j'\text{-}k}\right)\tilde{N}^{i'\text{-}t+k,j\text{-}k}_{i,j} + \tilde{N}^{i'\text{-}t+j,0}_{i,j} + N_{i\text{-}t,j} \\
&= \textstyle\sum_{k=1}^{j'} N_{i\text{-}t+k,j\text{-}k}\tilde{N}^{i'\text{-}t+k,j\text{-}k}_{i,j} + N_{i\text{-}t+j,0}\tilde{N}^{i'\text{-}t+j,0}_{i,j} + N_{i\text{-}t,j}\tilde{N}^{i'\text{-}t,j}_{i,j} = N'_{i,j} \,;
\end{aligned}$$

where the second the third equation involves changing dummies. When $i \le \hat{\sigma}$,

$$\begin{aligned}
N''_{i,j} &= \textstyle\sum_{k=\hat{\sigma}\text{-}t'}^{i\text{-}2} N_{k,T\text{-}t'\text{-}k}(\tilde{N}^{k+2,T\text{-}t'\text{-}k}_{i,j} + \tilde{N}^{k+1,T\text{-}t\text{-}k}_{i,j}) + N_{i\text{-}1,T\text{-}t\text{-}i} \\
&= \textstyle\sum_{k=\hat{\sigma}\text{-}t}^{i\text{-}1} N_{k\text{-}1,T\text{-}t\text{-}k}\tilde{N}^{k+1,T\text{-}t\text{-}k}_{i,j} + \sum_{k=\hat{\sigma}\text{-}t'}^{i\text{-}2} N_{k,T\text{-}t'\text{-}k}\tilde{N}^{k+1,T\text{-}t\text{-}k}_{i,j} + N_{i\text{-}1,T\text{-}t\text{-}i} \\
&= \textstyle\sum_{k=\hat{\sigma}\text{-}t}^{i\text{-}2} (N_{k\text{-}1,T\text{-}t\text{-}k} + N_{k,T\text{-}t'\text{-}k})\tilde{N}^{k+1,T\text{-}t\text{-}k}_{i,j} + N_{i\text{-}2,T'\text{-}t\text{-}i} + N_{\hat{\sigma}\text{-}t',T\text{-}\hat{\sigma}}\tilde{N}^{\hat{\sigma}\text{-}t,T'\text{-}\hat{\sigma}}_{i,j} + N_{i\text{-}1,T\text{-}t\text{-}i} \\
&= \textstyle\sum_{k=\hat{\sigma}\text{-}t}^{i\text{-}2} N_{k,T\text{-}t\text{-}k}\tilde{N}^{k+1,T\text{-}t\text{-}k}_{i,j} + N_{i\text{-}1,T'\text{-}t\text{-}i} = N'_{i,j} \,,
\end{aligned} \tag{26}$$

where the fourth equality in (26) uses (8) and the fact that $\tilde{N}^{\hat{\sigma}-t,T-\hat{\sigma}+1}_{i,j} = 0$. Therefore in this case, $N'_{i,j} = N''_{i,j}$ for any (i, j) and thus $\tilde{f}_p(H; I, T | s_{-t'} = 1) = \tilde{f}_p(H; I, T | s_{-t} = 1)$.

Case 1-ii. $\hat{\sigma}' = \hat{\sigma} + 1$, in this case $(\hat{\sigma} - t - 1, T - \hat{\sigma}) \notin \Omega_H(I, T)$. In this case, for $i \ge \hat{\sigma}'$ still we have $N'_{i,j} = N''_{i,j}$. For $i \le \hat{\sigma}$,

$$\begin{aligned}
N''_{i,j} &= \textstyle\sum_{k=\hat{\sigma}\text{-}t}^{i\text{-}2} N_{k,T\text{-}t'\text{-}k}(\tilde{N}^{k+2,T\text{-}t'\text{-}k}_{i,j} + \tilde{N}^{k+1,T\text{-}t\text{-}k}_{i,j}) + N_{i\text{-}1,T\text{-}t\text{-}i} \\
&= \textstyle\sum_{k=\hat{\sigma}'\text{-}t}^{i\text{-}1} N_{k\text{-}1,T\text{-}t\text{-}k}\tilde{N}^{k+1,T\text{-}t\text{-}k}_{i,j} + \sum_{k=\hat{\sigma}\text{-}t}^{i\text{-}2} N_{k,T\text{-}t'\text{-}k}\tilde{N}^{k+1,T\text{-}t\text{-}k}_{i,j} + N_{i\text{-}1,T\text{-}t\text{-}i} \\
&= \textstyle\sum_{k=\hat{\sigma}'\text{-}t}^{i\text{-}2} N_{k,T\text{-}t\text{-}k}\tilde{N}^{k+1,T\text{-}t\text{-}k}_{i,j} + N_{i\text{-}1,T\text{-}t\text{-}i+1} + N_{\hat{\sigma}\text{-}t,T\text{-}\hat{\sigma}'}\tilde{N}^{\hat{\sigma}'\text{-}t,T\text{-}\hat{\sigma}}_{i,j} \\
&= \textstyle\sum_{k=\hat{\sigma}'\text{-}t}^{i\text{-}1} N_{k,T\text{-}t\text{-}k}\tilde{N}^{k+1,T\text{-}t\text{-}k}_{i,j} + N_{\hat{\sigma}\text{-}t,T\text{-}\hat{\sigma}'}\tilde{N}^{\hat{\sigma}'\text{-}t,T\text{-}\hat{\sigma}}_{i,j} \\
&= N'_{i,j} - N_{\hat{\sigma}\text{-}t,T\text{-}\hat{\sigma}}\tilde{N}^{\hat{\sigma}'\text{-}t,T\text{-}\hat{\sigma}}_{i,j} + N_{\hat{\sigma}\text{-}t,T\text{-}\hat{\sigma}'}\tilde{N}^{\hat{\sigma}'\text{-}t,T\text{-}\hat{\sigma}}_{i,j} \\
&= N'_{i,j} - N_{\hat{\sigma}\text{-}t\text{-}1,T\text{-}\hat{\sigma}}\tilde{N}^{\hat{\sigma}'\text{-}t,T\text{-}\hat{\sigma}}_{i,j} = N'_{i,j} \,,
\end{aligned}$$

where the last equality used the fact that $N_{\hat{\sigma}-t-1,T-\hat{\sigma}} = 0$ Therefore in this case, $N'_{i,j} = N''_{i,j}$ for any (i,j) and thus $\tilde{f}_p(H;I,T|s_{-t'}=1) = \tilde{f}_p(H;I,T|s_{-t}=1)$.

Case 2. Similarly, if $s_{-t} = -1$, let $N'_{i,j}$ (resp. $N''_{i,j}$) be the number of staircase walks from $(0,0)$ to (i,j) that contains the step from (σ,φ) to $(\sigma,\varphi+1)$ for any $\sigma+\varphi = T-t$ (resp. $\sigma+\varphi = T-t'$). Thus

$$\begin{aligned}\tilde{f}_p(H;I,T|s_{-t}=-1) &= p^{\sigma(H)}(1-p)^{\varphi(H)}\textstyle\sum_{i=\hat{\sigma}}^{T} N'_{i,T-i}p^i(1-p)^{T-i}\ ;\\ \tilde{f}_p(H;I,T|s_{-t'}=-1) &= p^{\sigma(H)}(1-p)^{\varphi(H)}\textstyle\sum_{i=\hat{\sigma}}^{T} N''_{i,T-i}p^i(1-p)^{T-i}\ .\end{aligned}$$

Similar to (24), we have $N'_{i,j} = \sum_{k=1}^{j} N_{i-z+k,j-k}\tilde{N}_{i,j}^{i-z+k,j+1-k}$ for $z = t,t'$. For simplicity, I do not list the specific subcases as in Case 1. But all equations hold allowing $N_{i,j} = 0$ if $(i,j) \notin \Omega_H(I,T)$ and $\tilde{N}_{i,j}^{\sigma,\varphi} = 0$ if $\sigma > i$ or $\varphi > j$.

$$\begin{aligned}N''_{i,j} &= \textstyle\sum_{k=1}^{j} N_{i-t'+k,j-k}(\tilde{N}_{i,j}^{i-t+k,j+1-k} + \tilde{N}_{i,j}^{i-t'+k,j+2-k})\\ &= \textstyle\sum_{k=1}^{j} N_{i-t'+k,j-k}\tilde{N}_{i,j}^{i-t+k,j+1-k} + \sum_{k=0}^{j-1} N_{i-t+k,j-k-1}\tilde{N}_{i,j}^{i-t+k,j+1-k}\\ &= \textstyle\sum_{k=1}^{j-1}(N_{i-t'+k,j-k} + N_{i-t+k,j-k-1})\tilde{N}_{i,j}^{i-t+k,j+1-k} + \tilde{N}_{i,j}^{i-t+j,1}\\ &= \textstyle\sum_{k=1}^{j-1} N_{i-t+k,j-k}\tilde{N}_{i,j}^{i-t+k,j+1-k} + \tilde{N}_{i,j}^{i-t+j,1} = N'_{i,j}\ .\end{aligned}$$

Therefore $\tilde{f}_p(H;I,T|s_{-t'}=-1) = \tilde{f}_p(H;I,T|s_{-t}=-1)$.

References

ARCIDIACONO, P., E. AUCEJO, A. MAUREL, AND T. RANSOM (2016): "College attrition and the dynamics of information revelation," Tech. rep., National Bureau of Economic Research.

AVAL, J.-C. (2008): "Multivariate Fuss–Catalan numbers," *Discrete Mathematics*, 308, 4660–4669.

BANERJEE, A. V. (1992): "A simple model of herd behavior," *The quarterly journal of economics*, 107, 797–817.

BANKS, J. S. AND R. K. SUNDARAM (1992): "A class of bandit problems yielding myopic optimal strategies," *Journal of Applied Probability*, 29, 625–632.

BEN-PORATH, E., E. DEKEL, AND B. L. LIPMAN (2018): "Disclosure and choice," *The Review of Economic Studies*, 85, 1471–1501.

BENJAMIN, D. J. (2019): "Errors in probabilistic reasoning and judgment biases," in *Handbook of Behavioral Economics: Applications and Foundations*, Elsevier, vol. 2, 69–186.

BERGGREN, D. (2017): "IPO Underpricing and techvaluation An empirical study of the Swedish IPO market," .

BERKSON, J. (1946): "Limitations of the application of fourfold table analysis to hospital data," *Biometrics Bulletin*, 2, 47–53.

BIKHCHANDANI, S., D. HIRSHLEIFER, AND I. WELCH (1992): "A theory of fads, fashion, custom, and cultural change as informational cascades," *Journal of political Economy*, 100, 992–1026.

BOLTON, P. AND C. HARRIS (1999): "Strategic experimentation," *Econometrica*, 67, 349–374.

CHAN, K., J. WANG, AND K. J. WEI (2004): "Underpricing and long-term performance of IPOs in China," *Journal of Corporate Finance*, 10, 409–430.

COVER, T. AND M. HELLMAN (1970): "The two-armed-bandit problem with time-invariant finite memory," *IEEE Transactions on Information Theory*, 16, 185–195.

DONG, M. (2018): "Strategic experimentation with asymmetric information," *Penn State University, https://sites. google. com/site/miaomiaodong0/1-delimit*.

ERCAN, M., S. ÇEVIKEL, ET AL. (2011): "Survey on IPO underpricing in the Istanbul Stock Exchange," *Iktisat Isletme ve Finans*, 26, 85–117.

ESPONDA, I. (2008): "Behavioral equilibrium in economies with adverse selection," *American Economic Review*, 98, 1269–91.

ESPONDA, I. AND D. POUZO (2017): "Conditional Retrospective Voting in Large Elections," *American Economic Journal: Microeconomics*, 9, 54–75.

FRISTEDT, B. AND D. A. BERRY (1988): "Optimality of myopic stopping times for geometric discounting," *Journal of applied probability*, 25, 437–443.

GITTINS, J. C. (1979): "Bandit processes and dynamic allocation indices," *Journal of the Royal Statistical Society. Series B (Methodological)*, 148–177.

HALAC, M. AND I. KREMER (2020): "Experimenting with career concerns," *American Economic Journal: Microeconomics*, 12, 260–88.

HE, K. (2018): "Mislearning from Censored Data: The Gambler's Fallacy in Optimal-Stopping Problems," *arXiv preprint arXiv:1803.08170*.

HECKMAN, J. J. (1979): "Sample selection bias as a specification error," *Econometrica: Journal of the econometric society*, 153–161.

JEHIEL, P. (2018): "Investment strategy and selection bias: An equilibrium perspective on overoptimism," *American Economic Review*, 108, 1582–97.

LEIGHTON, F. AND R. RIVEST (1986): "Estimating a probability using finite memory," *IEEE Transactions on Information Theory*, 32, 733–742.

MIKLÓS-THAL, J., M. RAITH, AND M. SELOVE (2018): "What Are We Really Good At? Product Strategy with Uncertain Capabilities," *Marketing Science*, 37, 294–309.

MILLER, J. B. AND A. SANJURJO (2018): "Surprised by the hot hand fallacy? A truth in the law of small numbers," *Econometrica*, 86, 2019–2047.

MOSCARINI, G. AND L. SMITH (2001): "The Optimal Level of Experimentation," *Econometrica*, 69, 1629–1644.

ROSENBERG, D., E. SOLAN, AND N. VIEILLE (2007): "Social learning in one-arm bandit problems," *Econometrica*, 75, 1591–1611.

SETHI, R. AND M. YILDIZ (2016): "Communication with unknown perspectives," *Econometrica*, 84, 2029–2069.

SMITH, L., P. SØRENSEN, AND J. TIAN (2017): "Informational herding, optimal experimentation, and contrarianism," *Revised and resubmitted for Review of Economic Studies*.

THOMAS, C. (2019): "Experimentation with reputation concerns–Dynamic signalling with changing types," *Journal of Economic Theory*, 179, 366–415.

TVERSKY, A. AND D. KAHNEMAN (1973): "Availability: A heuristic for judging frequency and probability," *Cognitive psychology*, 5, 207–232.

WALD, A. (1947): "Foundations of a general theory of sequential decision functions," *Econometrica, Journal of the Econometric Society*, 279–313.

WILSON, A. (2014): "Bounded memory and biases in information processing," *Econometrica*, 82, 2257–2294.